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JULY, 1861.

EDITED BY
J. D. RUNKLE, A.M., A.A.S.

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THE MATHEMATICAL MONTHLY.

The excellent notices which the Monthly continues to receive from both sides of the Atlantic must be a source of great encouragement to all its friends. The *Lady's and Gentleman's Diary* for 1861, edited by the eminent mathematician, PROF. W. S. B. WOOLHOUSE, contains the following: —

"This American periodical, which has now completed its second volume, continues to be most ably conducted, and is well supplied by talented correspondents. Besides numerous mathematical questions and solutions, it comprises original papers of considerable variety and sterling interest, and its publication must contribute largely to the diffusion and advancement of mathematical science."

A recent number of the *Boston Daily Transcript* gives the following item: —

"AMERICAN LITERATURE ABROAD. — Fourteen columns of the last Athenæum are devoted to two American authors, — Motley and Emerson. That paper which rarely commends anything from this side of the Atlantic, gives 'The Mathematical Monthly' the following first-rate notice. That magazine is published at Cambridge, and edited by Mr. J. D. Runkle: —

"'This is a collection of simple things for comparative elementary students. Our English mathematical journals are very lofty, and the beginner has no part in them. This American journal is full of pleasing elementary matter, and contains some things worthy the attention of the finished mathematician. We should like to see the same sort of thing in England; but where are we to find the editor? He must be of high knowledge and elementary turn combined, with firmness enough to keep his journal down to the intelligible point by refusal of contributions of too lofty a character, and skill enough to keep it up to the interesting point by selecting from the loads of lower learning with which he would be inundated.'"

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The Mathematical Monthly.

IN commencing our third volume, we beg to assure the friends of this Journal, that all legitimate efforts will be made to sustain it, together with the Prizes, which have proved such an excellent feature of its plan. It will be seen (see 2d page of cover) that the amount of the Prizes has been increased. But all efforts on our part must prove unavailing if the subscription list, with other sources of income, will not cover the expenses which we are incurring to make the Monthly really valuable and attractive to all interested in it. To show our friends at home what is thought of the Monthly abroad, we venture to make short extracts from letters lately received.

"The work appears to me so excellent, that I have at once ordered the series for the Library of the Observatory.

"G. B. AIRY, *Ast. Royal.*"

"*Royal Obs., Greenwich, July* 5 1860.

"I have looked through it with great interest, and especially because it shows to what an extent interest in mathematical subjects is felt through the United States. I do not believe that it would be possible in England to get up a Mathematical periodical brought out in so handsome a form and supported by so large a number of subscribers.

"*Trinity College, Dublin, July* 12, 1860. GEORGE SALMON."

"DEAR SIR. — I thank you sincerely for the copy you so kindly sent me of the first volume of the Mathematical Monthly We have not, that I am aware of, any work of a similar character in England. I feel confident it will serve a valuable purpose in stimulating and giving a healthy tone to the study of Mathematics in the higher class of schools, and judging from this volume, it will be a work of as great interest and give as much matter for thought to the tutor as to the student Original investigations on the mere elementary branches of science are perhaps not often to be expected: those portions have now been so long and so frequently considered, and by so many different minds, that something new — a really original idea — appears only at very distant intervals. Still many interesting questions may, and, as your first volume shows, do arise even in the earliest branches of Mathematical science; whether discussions on the principles and on the best methods of placing them before the pupil, or examples and problems which in their solutions show a large amount of ingenuity and mathematical skill. The Mathematical Monthly claims a special interest from the very fact of its character rendering it useful to the many without detracting from its scientific merit.

"I remain, your obedient servant.

"*St. John's College, Cambridge, July* 26, 1860. HUGH GODFRAY."

We feel confident that a little effort on the part of our friends would double our subscription list. See terms to clubs and new subscribers taking Vols. I., II., III. We could then engrave some of the fine portraits of LAGRANGE, MONGE, CAUCHY, BIOT, &c., lately received; and, indeed, incur any expense to increase the value of the Monthly to all Will our friends make up their clubs at once, and notify us, so that we can decide upon the size of our edition?

THE

MATHEMATICAL MONTHLY.

Vol. III. . . . JULY, 1861. . . . No. X.

THIRD PRIZE ESSAY.—THE METHOD OF PROJECTIONS.*

By Arthur W. Wright, New Haven, Conn.

1. If from any point in space straight lines be drawn through the angular points of any figure, and upon each of the lines so drawn any point be chosen, the lines joining such points will form a new figure bearing a certain relation to the primitive figure, and called a *projection* of it. If the ratios of the segments intercepted by the points of the two figures are equal, and of the same algebraic sign, the figures will be similar, and the ratio of their perimeters the

* The first use of Projection seems to have been made as early as, or earlier than, the time of Hipparchus, who employed it in the construction of geographical charts and star maps. Long after him, Claudius Ptolemæus employed for the same purpose the process which has more recently received the name of *Stereographic Projection.* The method of projections in *geometry,* however, is of comparatively recent date. The fact that the projections of the circle upon different planes are conic sections, has, indeed, long been known, and it is probable that the ancients were not wholly unacquainted with it. Newton, in his *Opuscula Mathematica, Philosophica, et Philologica,* Vol. I. p. 264, mentions the fact that the shadow of a circle thrown upon a plane is a conic section, and the shadow of any curve a curve of the same degree. De la Hire also, who was probably the first to employ in geometrical investigations the points afterwards called by Servois *poles*, and the lines named *polars* by Gergonne, seems to have been acquainted with the fact that all the conics may be obtained from the circle by projection. Little or no use, however, was made of this principle in geometry until the early part of the seventeenth century, when Desargues, and, after him, Pascal, applied it to the investigation of the properties of conic sections. Desargues was likewise the discoverer of the theory of

same as that of the segments of the projecting lines. The term *projection*, however, is usually limited in its application to the figure formed by joining the points in which a plane intersects the lines drawn from any point in space through the angular points of any *plane* figure, and it is in this sense that it will be employed in the present essay.

2. The point from which the projecting lines are drawn is called

---

the *involution of six points*, and PASCAL of the celebrated proposition which bears his name, respecting the properties of the hexagon inscribed in a conic.

MONGE, however, was the first to give prominence to the method of projections, and made it the basis of his *Géométrie Descriptive*, in which he investigates the properties of figures of three dimensions by means of their projections upon two planes intersecting at right angles. He also established more completely the theorems in regard to poles and polars of curves of the second degree, investigated the properties of the points called by him *centres of similitude* of circles, and proved that those of three circles lie three by three in straight lines, together with the analogous proposition concerning the centres of similitude of four spheres. After MONGE, COUSINERY, in his *Géométrie Perspective*, extended the method of projections, and simplified it by making use of a single projection upon a single plane; and also BRIANCHON, applying both DESARGUES'S theory of the involution of six points and the principles of projection, made many interesting discoveries, and wrote an admirable treatise, entitled *Mémoire sur les Lignes du Second Ordre*, to which PONCELET acknowledged himself greatly indebted. CARNOT developed the theory of *transversals*, and by a beautiful generalization succeeded in applying it to curved lines and surfaces.

It is to PONCELET, however, that the method of projections is most indebted for a systematic and full development, and his work, bearing the title *Traité des Propriétés Projective des Figures*, is a complete *résumé* of all that had been accomplished up to his time, together with many new and valuable propositions of his own. PONCELET introduced the *principle of continuity*, and was the first to make use of the doctrine of the *homology of figures*, as well as that of *reciprocal polars*, which he employed very elegantly and effectively. He also made use of *imaginary quantities* in geometry, a principle which has since received many important applications in the hands of M. CHASLES, who is likewise the author of many elegant theorems regarding the anharmonic ratio of four and the involution of six points, and has besides done much to extend and perfect the method of projections. We will only remark, before proceeding with the method itself, that although it has been objected that it is not in all respects sufficiently rigorous, yet its simplicity and beauty recommend it to all lovers of pure geometry, while as an instrument of investigation and discovery it is invaluable.

the *centre of projection*, and the plane upon which the projection is thrown is called the *plane of projection.*

3. The projection of a point is a point, since it is the intersection of the projecting line with the plane of projection.

4. The projection of a straight line is a straight line; for all the lines drawn from the centre of projection to such a line are necessarily in one plane, and its projection lies in the intersection of this plane with the plane of projection, and therefore is a straight line. Hence, if any number of straight or curved lines meet in one point or in any number of points, their projections will also meet in one point or in an equal number of points, as the case may be.

5. The projection of any curve is a curve of the same degree. For the degree of a curve is determined by the number of points in which it is met by a straight line. If, therefore, a curve of the nth degree is met by a straight line in n points, these points will be projected into n new points, in which the projection of the straight line meets that of the curve, which is therefore of the nth degree. Hence the projection of a circle, or any conic section, is another conic section. Also a chord of any curve meeting it in the points M and N will be projected into a chord of the projected curve meeting it in the points m and n. Now, when M and N coincide, m and n also coincide, the chord becomes a tangent, and its projection a tangent to the projected curve.

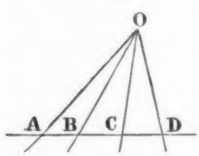
6. It is plain that any number of straight lines meeting in the same point may be projected into parallels, for, O being the centre of projection and S the common point of the given lines, if these lines are projected upon a plane parallel to a plane passing through O and S, the line $O\,S$ will be parallel to the plane of projection, and hence the projection of S upon the latter will be at infinity, and the projections of the lines drawn through S will be straight lines meeting in a point at infinity, and therefore parallel. Also, if there be

any number of straight lines meeting in n points in one straight line, lm, their projections upon a plane parallel to Olm will be n systems of parallels. Hence, on the contrary, the projections of parallel lines, that is, of lines meeting in a point at infinity, are in general lines meeting in a point at a finite distance. So, also, if we have in a plane n systems of parallel straight lines, their projections will in general be n systems of straight lines, meeting in n points upon one straight line at a finite distance.

7. It is evident, from the nature of projection, that its principal application is to the relations of *position* of figures and their parts, and not to those of *absolute magnitude.* Those properties which, being true for any figure, are true also for its projections, are called *projective properties.* Hence, if we wish to prove any projective theorem in regard to any figure, we have only to establish it for one of its simplest projections, and thence can infer its truth in regard to the primitive figure.

8. If now through any point, as O (Fig. 1), there be drawn four straight lines making any angle with one another, and any line, as $ABCD$, be drawn intersecting them, putting p for the perpendicular from O on AD, we shall have

Fig. 1.



$$AB \,.\, \tfrac{1}{2}p = \tfrac{1}{2} AO \,.\, OB \sin AOB,$$

$$\therefore \quad AB = AO \,.\, OB \sin AOB \div p;$$

and, similarly,

$$BC = BO \,.\, OC \sin BOC \div p,$$
$$CD = CO \,.\, OD \sin DOC \div p,$$
$$AD = AO \,.\, OD \sin AOD \div p,$$

$$\therefore \quad \frac{AB \,.\, CD}{BC \,.\, AD} = \frac{\sin AOB \sin DOC}{\sin AOD \sin BOC} = \text{constant}.$$

Now, since this expression does not involve the distances of A, B,

C, and *D* from *O*, but only the sines of the angles at that point, the relation in question is independent of the position of *A D*, and therefore holds true for any of its projections. Hence, if from the points *A*, *B*, *C*, and *D* straight lines be drawn to any point in space, the segments made by those lines upon any line intersecting them will be in the same ratio as the above. This property is therefore projective.

9. The line *A D* is called a *transversal*, and the ratio $\frac{AB.CD}{BC.AD} = \text{const.}$, the *anharmonic* ratio of the points *A*, *B*, *C*, and *D*, in distinction from the *harmonic* ratio, which is the name given to this ratio in the particular case in which it is equal to 1. For if

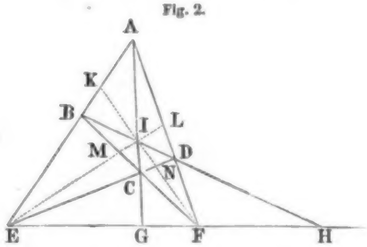
$$\frac{AB.CD}{BC.AD} = 1, \qquad \frac{AB}{BC} = \frac{AD - BD}{BD - CD} = \frac{AD}{CD},$$

or, $$AD : CD :: AD - BD : BD - CD,$$

which is a harmonical proportion. Now, if *D* is at an infinite distance, *AD* and *CD* become ultimately equal, so that the anharmonic ratio becomes $\frac{AB}{BC}$ = constant, and the harmonic $\frac{AB}{BC} = 1$, or $AB = BC$, and hence *A C* is bisected in *B*.

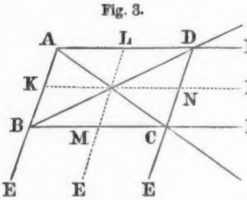
10. Again, let *A B C D* (Fig. 2) be a quadrilateral, of which the opposite sides, *A B* and *C D*, *A D* and *B C*, meet in the points *E* and *F* respectively, forming a complete quadrilateral, as it is called. Draw the diagonals *E F*, *B D*, and *A C*, the two latter meeting *E F* in *G* and *H*, and *E I* and *F I* meeting the sides of the figure in *M*, *L*, *K*, and *N*. The figure thus formed is composed of nine lines, each divided harmonically in four points. For if the figure is projected in such a manner that the line *E H* passes to infinity, the lines passing through *E* and *F* become parallels (6), and

Fig. 2.



the quadrilateral $ABCD$ becomes a parallelogram, with the lines LM and KN drawn through the intersection of its diagonals parallel to the sides of the figure, as in Fig. 3. Hence each of the lines of the figure is divided in three points, of which the middle one in each case bisects the distance between the other two. The three points on each line are therefore three points of a harmonic group, of which the fourth is at infinity on the projection of EH (Fig. 2), (9). This being true for Fig. 3, is therefore true for Fig. 2, of which it is a projection. Again, the four straight lines IE, IG, IF, and IH (Fig. 2) being drawn from the same point, I, through the four harmonic points, E, C, N, and D, form a harmonic pencil, and therefore (8) cut the line EH harmonically in the four points E, G, F, and H, so that we have this theorem.

Fig. 3.



11. Each of the three diagonals of a complete quadrilateral is cut harmonically by the other two.

12. An interesting application of this theorem is made in the solution of the problem, to find, with the rule alone, the distance of an inaccessible point, as H (Fig. 2), from a given point, F. For, taking any other point, as E in the line FH, we have only to construct a complete quadrilateral whose opposite sides pass through the points E and F, and one of its diagonals through H. The other diagonal will divide the line EH harmonically (11), giving us the ratio $\frac{EG}{GF} = \frac{EH}{FH}$, which may be written $EG : GF :: EF + FH : FH$, whence we have

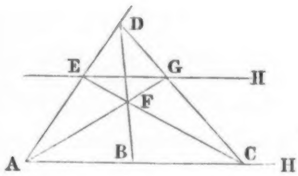
$$EG - GF : GF :: EF : FH;$$

therefore, the first three terms being known, the fourth, FH, may be found.

13. Another application may be made in the solution of the

problem, having given the middle point, B, of a finite straight line, AC, to draw with the rule alone a straight line passing through a given point, E, and parallel to AC. Join AE (Fig. 4) and EC, and from B draw a line intersecting these lines in any points, D and F; draw CD and AFG, and join EG. The figure thus formed is a complete quadrilateral, and the diagonals DF and EG cut the diagonal AC harmonically (11). But the point B bisects the line AC; therefore the fourth harmonic point, H, is at infinity (9), and hence EG is parallel to AC.

Fig. 4.



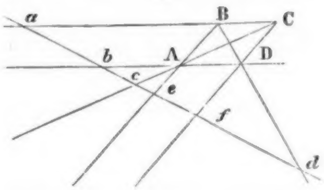
14. Again, in Fig. 3 the lines LE and KF determine eight segments upon the four sides of the parallelogram, and we have the following equation from the equality of the segments: —

$$AK.BM.CN.DL = AL.DN.CM.BK.$$

Now since for each line, AK, &c., we have, as in (8), a term of the form $AO.OK \sin AOK \div p$, it is evident from its symmetry that the equation reduces to one expressing the relation of the angles between the lines drawn from any point, O, and the points A, B, C, D, K, L, M, and N. Hence this property is projective, and therefore is true for any quadrilateral.

15. Let the transversal ad in Fig 5 meet the pairs of sides and the two diagonals of the parallelogram $ABCD$ in the points ab, ef, and cd respectively; we have, by similar triangles,

Fig. 5.



$$\frac{af}{ac}=\frac{be}{bc},\ \frac{ae}{ad}=\frac{bf}{bd},\ \frac{ea}{ed}=\frac{fb}{fd},\ \text{and}\ \frac{eb}{ec}=\frac{fa}{fc},$$

from which we have the equations

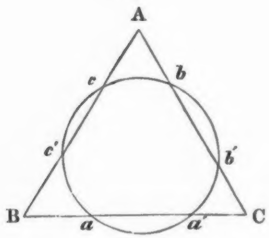
$$\frac{ae.af}{ac.ad}=\frac{be.bf}{bc.bd}, \qquad \text{and} \qquad \frac{ea.eb}{ec.ed}=\frac{fa.fb}{fc.fd}.$$

These relations are projective (14), and hence are true for the quadrilateral of which Fig. 5 may be considered as the projection. Now considering two of the opposite sides of this quadrilateral as the diagonals of the new quadrilateral formed by the two other sides and the diagonals, we have in a similar manner the equation $\frac{ca\,.\,cb}{ce\,.\,cf}=\frac{da\,.\,db}{de\,.\,df}$. Combining the three equations, we have the four following relations: —

$$cb\,.\,ea\,.\,fd = da\,.\,ec\,.\,fb \qquad cf\,.\,db\,.\,ea = bf\,.\,ca\,.\,de,$$
$$cb\,.\,ed\,.\,fa = da\,.\,eb\,.\,fc, \qquad ca\,.\,df\,.\,eb = db\,.\,ec\,.\,fa,$$

which are projective (14). The six points a, b, c, d, e, and f are said to be in *involution*. Hence the theorem, any quadrilateral with its diagonals is cut by a transversal in six points in involution.

Fig. 6.



16. Again, if the sides of the triangle ABC (Fig. 6) are cut by a circle in the points a, a', b, b', c, c', we have (Euclid, 3. 35 and 36) the equations

$$\frac{Ac\,.\,Ac'}{Ab\,.\,Ab'}=\frac{Ba\,.\,Ba'}{Bc\,.\,Bc'}=\frac{Cb\,.\,Cb'}{Ca\,.\,Ca'}=1,$$

which, multiplied together, give the equation

$$Ac\,.\,Ac'\,.\,Ba\,.\,Ba'\,.\,Cb\,.\,Cb' = Ab\,.\,Ab'\,.\,Bc\,.\,Bc'\,.\,Ca\,.\,Ca',$$

which is projective in its nature (14), so that we have Carnot's theorem, if a conic meets the sides BC, CA, and AB of a triangle, ABC, in the points a, a', b, b', and c, c' respectively, then

$$Ac\,.\,Ac'\,.\,Ba\,.\,Ba'\,.\,Cb\,.\,Cb' = Ab\,.\,Ab'\,.\,Bc\,.\,Bc'\,.\,Ca\,.\,Ca'.$$

17. If the conic, instead of intersecting the sides of the triangle, is inscribed within it, the points a, b, and c (Fig. 7) coincide with

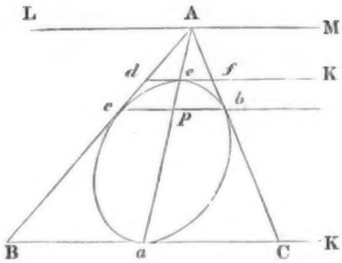
the points a'. b', and c' respectively, and the above equation becomes

$$Ac \,.\, Ba \,.\, Cb = Ab \,.\, Bc \,.\, Ca, \qquad (a)$$

which is likewise true for all projections of the figure.

18. If now the side BC (Fig. 7) is parallel to cb, which is called the *chord of contact* of AB and AC, we have, by similar triangles, $Ac : Bc :: Ab : Cb$, or $Ac \,.\, Cb = Bc \,.\, Ab$; and therefore, from equation (a), $Ba = Ca$, so that if Ap is drawn through the middle of the chord of contact cb, it will pass through the point a. In the same manner it may be shown that, if the tangent df is drawn parallel to cb, it is bisected by Apa. Now, as this must hold true for every chord of contact parallel to cb, we have the theorem, that the middle points of parallel chords of conic sections lie in the same straight line with the point of intersection of tangents drawn through their extremities, or, since this line is evidently a diameter of the curve, the chords of contact of tangents drawn through a given point are bisected by the diameter passing through that point, and are conjugate to it.

Fig 7.



19. Again, in Fig. 7, the three tangents BC, Bd, and dK may be considered as forming a triangle circumscribing the conic, and having its vertex, K, at infinity. Hence (17) $Bc \,.\, de \,.\, Ka = cd \,.\, Ba \,.\, Kc$. But $Ka = Ke$ ultimately. Therefore $Bc \,.\, de = cd \,.\, Ba$, or $\frac{Bc}{cd} = \frac{Ba}{de}$, and since BC, bc, and df are parallel, $\frac{Bc}{cd} = \frac{pa}{pe}$, and $\frac{Ba}{de} = \frac{Aa}{Ae}$; therefore $\frac{pa}{pe} = \frac{Aa}{Ae}$, which is a harmonical ratio (9). Therefore the diameter drawn through the intersecting point of two tangents to a conic is cut harmonically by the curve and the chord of contact.

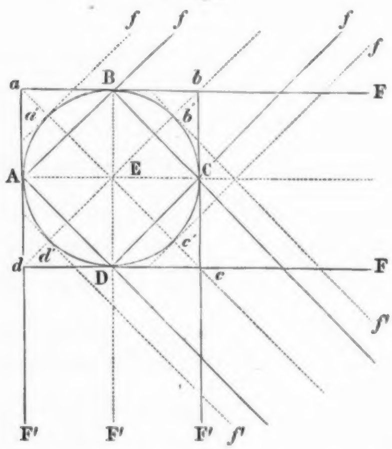
20. If now one of the extremities, a, for instance, of the diameter

of a conic passes to infinity, as is the case in the parabola, the equation $\frac{pa}{pe} = \frac{Aa}{Ae}$ becomes ultimately $Pe = Ae$, that is, in a parabola the part of the diameter lying between the summit of the circumscribed angle and the chord of contact is bisected by the curve.

21. If in Fig. 7 we draw through A a straight line to LM parallel to cb, its direction will be conjugate to that of the diameter Aa. This line is called the *polar* of the point p, which, on the other hand, is called the *pole* of the line LM. In the same manner the line cb is called the polar of the point A.

22. If now, in Fig. 8, we suppose a rectangle, $ABCD$, to be inscribed in the circle, and a parallelogram, $abcd$, circumscribed about it, having its points of tangency at the angles of $ABCD$, and the tangents $a'f$, $b'f'$, $c'f$, and $d'f'$ drawn parallel to its sides, it is evident that the opposite sides of these parallelograms meet at infinity, that their diagonals all pass through the centre of the circle, and that those of each parallelogram are parallel to the sides of the other, and hence meet them, together with the parallel tangents, at infinity. Now if we have a quadrilateral inscribed in any conic, and a straight line, mn, passing through the intersections of the opposite sides of that quadrilateral, it is evident that we may project them upon another plane in such a manner that the conic will become a circle, while the straight line mn passes to infinity. For we may take O, the centre of projection, in such a manner that the plane Omn will be parallel to the plane making a circular section with the cone formed by lines

Fig. 8.



drawn from O to all points of the conic, in which case the line will be projected to infinity (6), and the conic become a circle. Fig. 8, then, may be regarded as the projection of a conic and of its inscribed and circumscribed quadrilaterals, the angles of the former being at the points of tangency of the latter, and the opposite sides of both, with the lines from which the parallel tangents in Fig. 8 were projected, meeting upon the same straight line (6). Hence we may derive the following important conclusions.

23. The four diagonals of the two quadrilaterals meet in one point, E, which is the pole of the line containing the four points of intersection of their opposite sides.

24. The diagonals of each quadrilateral pass through the two points where the opposite sides of the other intersect, of which points each one is the pole of the line drawn through the other and the point E; that is, it is the intersecting point of the tangents drawn through the intersection of that diagonal with the conic.

25. Every straight line drawn through the point E from any point in its polar is divided harmonically by this point and the conic.

26. The point E being evidently independent of the particular position of the quadrilateral, and changing only with a change of position of its polar, we have the proposition, that if from any point in a straight line in its plane two straight lines be drawn touching any conic, the chord of contact will pass through a fixed point. Hence, on the other hand, if a secant of any conic revolve about a fixed point, the tangents at its extremities will intersect on a fixed line, which solves Pr. Prob. No. V., p. 29, Vol. I., of the Mathematical Monthly.

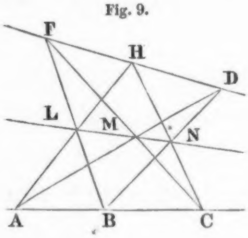
27. It is evident that the same method of proof may equally be applied to the case of the hexagon inscribed in a conic, and the circumscribed hexagon whose sides touch the conic at the angular points of the former, thus conducting us directly to the celebrated theorem of Pascal: —

The opposite sides of every hexagon inscribed in a conic meet in three points which lie in one straight line:

Also, in every hexagon circumscribed about a conic, the three diagonals joining the opposite angles meet in a point.

28. Now, since any two straight lines lying in the same plane may be considered as a conic section,* we have the theorem, If on each of any two straight lines three points, F, H, and D (Fig. 9), A, B, and C are taken, the three pairs of lines FB, AH; FC, AD; and HC, BD intersect in three points which lie in one straight line. This proves the construction given in the Mathematical Monthly, Vol. II. p. 266, Fig. 2.

Fig. 9.



29. The above proposition (27) also enables us to solve the problem, Having given five points of any conic section, to describe that conic section by points. For if the given points are A, B, C, D, and E, join AB, BC, CD, and DE, and let AB and DE meet in m. Through m draw any straight line meeting BC and CD in l and n respectively. Join nA and lE, meeting in F. F is evidently a point of the conic (27); and in the same manner we may obtain any number of points, and thus wholly determine the curve.

30. We will now briefly consider the case in which the centre of projection is at infinity, and the projecting lines perpendicular to the plane of projection, to designate which the term *orthogonal projection* is sometimes employed. The projecting lines are evidently parallel, and we may notice the following particulars.

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* That two straight lines may be considered as a conic section is evident: 1st, from the fact that a straight line intersects two straight lines in two, and only two points (see above, Sec. 5); 2d, considering the conic as formed by the intersection of a cone and a plane, the conic will become two straight lines when the plane passes through the axis of the cone; and, 3d, because the analytical equation representing two straight lines is of the second degree (SALMON's Conic Sections, Sec. 67 *et seq.*).

31. Parallel lines are projected into parallel lines, and are diminished in a constant ratio.

32. Lines parallel to the line of intersection of the plane of a figure and that of its projection, are unaltered by projection, while those that are at right angles to this line are diminished in the ratio of the secant of the angle of inclination of the planes. So also the areas of plane figures are diminished by projection in the same ratio.

33. Now, if in any circle we draw a diameter parallel to the intersecting line of its plane with that of projection, and draw lines at right angles to this diameter meeting the circumference, these lines will all be diminished in a constant ratio (31), and hence the resulting figure will be an ellipse when the whole is projected upon another plane.

This property, therefore, affords a ready solution of the following problem:* If the three angles A, B, and C of a parallelogram, $ABCD$, are in an ellipse, and the sides DA and DC meet the curve in E and F, then will a tangent to the curve at B be parallel to a line drawn from E to F; for the truth of the proposition is evident in the case of the circle from the symmetry of the figure, and hence (31, 33) it is equally true for the ellipse.†

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* Prize Problem proposed to the Sophomore Class, Yale College, July, 1857.

† The titles of some of the most valuable works, either devoted to or containing chapters upon the method of projections, are given in the following list, together with those of some of the principal journals in which may be found papers upon the same subject.

PONCELET, *Propriétés Projective des Figures;* BRIANCHON, *Mémoire sur les Lignes du Second Ordre;* MONGE, *Géométrie Descriptive;* CARNOT, *Géométrie de Position, Essai sur la Théorie des Transversals;* PLÜCKER, *Analytisch-Geometrische Entwicklungen;* CHASLES, *Aperçu Historique sur l'Origin et le Développement des Methodes en Géométrie,* also his memoirs upon *Cones* and *Spherical Conics;* SALMON, *Conic Sections, Higher Plane Curves;* QUETELET'S *Correspondance;* GERGONNE'S *Annales des Mathématiques;* LIOUVILLE'S *Journal;* GRUNERT'S *Archiv für Mathematik;* CRELLE'S *Journal;* TERQUEM *et* GERONO, *Nouvelles Annales de Mathématiques.*

MATHEMATICAL NOTES.

By MATTHEW COLLINS, A. B., Senior Moderator in Mathematics and Physics of Trinity College, Dublin.

I. *New Geometrical Theorem, and its Elementary Demonstration.*

LEMMA. — Project a straight line, $B\,D\,C$,* upon any other straight line, $b\,c$, by the perpendiculars $B\,b$, $C c$, and $D\,d$; then if D lie between B and C, $D\,d \times B\,C$ will be $= B\,b \times C\,D + C\,c \times B\,D$.

For, draw $b'\,D\,c'$ through D parallel to $b\,c$, meeting the perpendiculars in b' and c'; then, as triangles $B\,D\,b'$ and $C\,D\,c'$ are obviously similar, $\therefore\ C\,D \times B\,b' = B\,D \times C\,c'$; that is,

$$C\,D\,(B\,b - D\,d) = B\,D\,(D\,d - C\,c),$$

and therefore, by transposition,

$$B\,b \times C\,D + C\,c \times B\,D = D\,d \times C\,D + D\,d \times B\,D,$$

therefore $= D\,d \times B\,C$.

Now to apply this lemma. Suppose $A\,D$ bisects $\angle B\,A\,C$, then, $B\,D$, $D\,C$ and $B\,C$ are as $A\,B$, $A\,C$, and $A\,B + A\,C$, and therefore by the lemma ($a\,b$ being *any* straight line upon which the points A, B, C, D, O are projected), $AB \times Cc + AC \times Bb = (A\,B + AC)\,D\,d$. Again, O being *any* point in AD, $AO \times Dd + DO \times Aa = AD \times Oa$. Now, if $C\,O$ bisect $\angle A\,C\,D$, O will then be the centre of the inscribed circle, and $A\,O : O\,D = A\,C : C\,D$, or $= A\,B : B\,D$, therefore also $= A\,B + A\,C : B\,C$; so that $O\,D$, $O\,A$, and $A\,D$ are as $B\,C$, $A\,B + A\,C$, and $A\,B + AC + B\,C$, and therefore by the lemma $(A\,B + AC + BC) \times O\,o = BC \times A\,a + (AB + AC)\,D\,d$, and as this last term $(A\,B + AC)\,D\,d$ was already proved

$$= A\,B \times Cc + AC \times B\,b;$$

* The figures will be readily supplied by the reader. — ED.

hence, then,

$$AB \times Cc + AC \times Bb + BC \times Aa = (AB + AC + BC)\, Oo.$$

As a particular case of this general theorem, let us now suppose that the arbitrary line ab touches the inscribed circle, then $Oo =$ radius, and $\therefore (AB + AC + BC)\, Oo = 2 \triangle ABC$. Whence this remarkable theorem,* viz.: If each side of a *fixed* triangle described about a circle be multiplied by the perpendicular let fall from its opposite vertex upon any *variable* tangent to the circle, the algebraic sum of the products will be constant and equal to twice the area of the triangle, considering, as usual, the two perpendiculars upon one side of the tangent as positive, and the single perpendicular on the opposite side as negative.

Corollary. — When ab passes through O, then Oo vanishes; whence this other theorem, viz.: If each side of a triangle be multiplied by the perpendicular distance of the opposite vertex from *any* diameter of its inscribed circle, the greatest of the three products will be equal to the sum of the other two.

II. *New and Easy Investigation of a Remarkable Equation in Spherical Trigonometry.*

Let ABC be a spherical triangle, whose angles are denoted by A, B, C, the sides respectively opposite to them being, as usual, denoted by a, b, c; produce AC to D, making $AD = 90°$. Join BD by an arc of a great circle; then, by the first or fundamental formula in spherical trigonometry applied to $\triangle BCD$,

$$\cos BD = \cos BC \cos CD + \sin BC \sin CD \cos BCD;$$

* An entirely different geometrical demonstration of the foregoing theorem has been inserted by me in an Irish Mathematical Almanac for the Year (1861), published by Purdon Brothers, 23 Bachelor's Walk, Dublin.

that is, $= \cos a \sin b - \sin a \cos b \cos C$. Again, by applying the same fundamental formula to $\triangle ABD$, we obtain

$$\cos BD = \cos AB \cos AD + \sin AB \sin AD \cos A;$$

that is, $= \sin c \cos A$, for $\cos AD = 0$ and $\sin AD = 1$, since $AD = 90°$; now, by equating these two values of $\cos BD$, dividing them by $\sin a$, and remembering that $\frac{\sin c}{\sin a} = \frac{\sin C}{\sin A}$, as is *easily proved geometrically* (see HYMER's Trigonometry, 3d edition, page 174), we obtain the required equation,

$$\cot a \sin b = \cot A \sin C + \cos b \cos C,$$

which, by the relations of the polar triangle, leads only to a similar equation, and to nothing new.

III. *New and Remarkable Use of the Polar Triangle in discovering Formulas in Spherical Trigonometry.*

1. By the first fundamental formula of spherical trigonometry, $\cos a = \cos b \cos c + \sin b \sin c \cos A$;

$$\therefore \cos a \cos A = \cos b \cos c \cos A + \sin b \sin c \cos^2 A;$$

and as this formula is of course also true for the polar triangle, whose sides and angles are the supplements of the angles and sides of the primitive triangle which are respectively opposite to them,

$$\begin{aligned}\therefore \cos(\pi - A)\cos(\pi - a) = {} & \cos(\pi - B)\cos(\pi - C)\cos(\pi - a) \\ & + \sin(\pi - B)\sin(\pi - C)\cos^2(\pi - a);\end{aligned}$$

that is, $\cos A \cos a = -\cos B \cos C \cos a + \sin B \sin C \cos^2 a$. Now by equating both these values of $\cos a \cos A$, putting $1 - \sin^2 a$ for $\cos^2 a$, and remembering that $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$, we find

(1) $\sin B \sin C - \sin b \sin c = \cos b \cos c \cos A + \cos B \cos C \cos a$;

a remarkable formula first discovered by CAGNOLI.

2. Again, by the same fundamental formula,

$$\cos b \cos c = \cos a - \sin b \sin c \cos A.$$

Multiply this by $\cos B \cos C$,

$$\therefore \cos b \cos c \cos B \cos C = \cos a \cos B \cos C - \sin b \sin c \cos A \cos B \cos C;$$

this formula being polarized as directed above, viz. by putting $\pi - A$ for a, $\pi - a$ for A, &c., gives again

$$\cos B \cos C \cos b \cos c = -\cos A \cos b \cos c + \sin B \sin C \cos a \cos b \cos c.$$

Now by equating both these values of $\cos b \cos c \cos B \cos C$, we obtain the remarkable equation,

$$\left.\begin{array}{r}\cos a \cos b \cos c \sin B \sin C \\ + \cos A \cos B \cos C \sin b \sin c\end{array}\right\} = \left\{\begin{array}{l}\cos B \cos C \cos a \\ + \cos b \cos c \cos A,\end{array}\right.$$

therefore, by (1), $= \sin B \sin C - \sin b \sin c$, which gives, by transposition,

$$\sin B \sin C (1 - \cos a \cos b \cos c) = \sin b \sin c (1 + \cos A \cos B \cos C),$$

$$\text{or,} \quad \therefore \frac{1 + \cos A \cos B \cos C}{1 - \cos a \cos b \cos c} = \frac{\sin B \sin C}{\sin b \sin c}, \quad \therefore = \frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}.$$

Now by putting, as usual, $1 - \frac{a^2}{2r^2} + \frac{a^4}{2.3.4.r^4} -$ &c., for $\cos a$, and $\frac{a}{r} - \frac{a^3}{2.3r^3}$, &c., for $\sin a$; and then supposing r (the radius of the sphere) to be infinitely great, the foregoing equation becomes, for a plane triangle,

$$\frac{1 + \cos A \cos B \cos C}{\frac{1}{2}(a^2 + b^2 + c^2)} = \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2}.$$

But I shall not proceed further, as these two instances sufficiently explain the method to those who have more abilities and leisure to push this curious and novel plan of proceeding further.

IV. *New and Easy Proof of* Napier's *Rules.*

By admitting only the first or fundamental formula in spherical trigonometry, viz. $\cos c = \cos a \cos b + \sin a \sin b \cos C$, *all* the cases of these useful rules may be demonstrated as follows by a *single* diagram. For the above equation being also true in the polar triangle, whose sides and angles are the supplements of the angles and sides of the primitive triangle which are respectively opposite to them,

$$\therefore \cos(\pi - C) = \cos(\pi - A)\cos(\pi - B) \\ + \sin(\pi - A) \times \sin(\pi - B)\cos(\pi - c),$$

that is, $-\cos C = \cos A \cos B - \sin A \sin B \cos C$; and now, if the spherical triangle ABC be right-angled at C, that is, if $C = 90°$, these two equations give $\cos c = \cos a \cos b$, or $= \cot A \cot B$. Now produce CB and CA to D and F, making $CD = CF = 90°$, and as $\angle C$ is right, therefore D is pole of CF, and F is pole of CD; $\therefore AD = 90° = FB$, and hence the great circles, whose poles are A and B, must pass through D and F, and their point of intersection, E, will therefore be pole of AB; and thus each vertex of the spherical pentagon $ABDEF$ is pole of its opposite side; and as any arc drawn from the pole is perpendicular to the polar, therefore the angles at C', C'', &c., where the alternate sides meet, are all right. Moreover, as the distance of the poles of two great circles is equal to the angle at which they intersect, therefore each side of this spherical pentagon is equal to the external angle at the opposite vertex; thus, arc $DE = \angle BAC = \angle A$, arc $EF = \angle B$, $\angle AFC^{\text{IV}} = BD = 90 - a$, &c.; hence the five sides of our spherical pentagon are Napier's five circular parts (or rather their complements), viz. the hypothenuse AB, the angles A and $B =$ the arcs DE, EF, and the *complements* of the legs $AF = 90 - b$, and $BD = 90 - a$; now the equation

$$\cos c = \cos a \cos b, \qquad \text{or} \qquad = \cot A \cot B,$$

gives $\cos AB = \sin BD \sin AF$, or $= \cot DE \cot EF$; and as the angles C', C'', &c. are right, therefore this relation must manifestly hold true for *every* side; that is, cos of *any* side of the pentagon is equal to the product of sines of the two adjacent or conterminous sides, or equal to the product of the cotangents of the two remote sides. Now when any two parts are adjacent in the original triangle, ABC, they are remote or separated from each other in the pentagon, and *vice versa;* thus the angle A is adjacent to AB and AC in the triangle, but $DE = \angle A$ is separated from AB and $AF (= 90 - AC)$ in the pentagon. Again, AC and BC are adjacent in the triangle, (for the right angle C lying between them is none of the five circular parts, and is therefore considered to have no effect in separating two adjacent or remote parts,) but their complements, AF and BD, are separated from each other in the pentagon by AB, &c. By attending, then, to this circumstance, we obviously get Napier's rules in their *improved* form, viz.: —

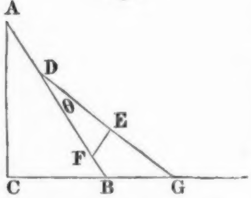
In a right-angled spherical triangle, the *cosine of the middle part is equal to the product of the cotangents of the two adjacent parts*, or *equal to the product of the sines of the two remote parts;* the five circular parts of the triangle in their consecutive order being $a' (= 90 - a)$, $b' (= 90 - b)$, A, c, B, a', b', &c., viz. the hypothenuse, the two oblique angles, and the complements of the two legs, the right angle being considered of no effect in separating any two parts, even when lying between them.

THE PENDULUM.*

By Thomas Sherwin, Principal, English High School, Boston.

Suppose a body, in virtue of gravity, to have moved along the inclined plane AB (Fig. 1) in any time to the point D, and to have acquired a final velocity per second $= V$. Suppose that at the point D the body meets another plane, DG, less elevated than AB, and making with AB the angle $BDG = \theta$. It is required to find the diminution of velocity which the body suffers at D in passing from one plane to the other.

Fig. 1.

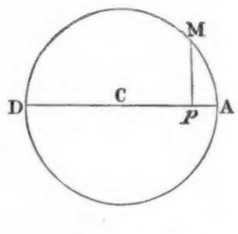


Take $DF =$ to the space over which the body would pass in one second, with the final velocity considered as uniform. DF then represents the final velocity V, or the force which produces this velocity. From F draw FE perpendicular to DG, and resolve the force DF into two forces or velocities, DE and EF. The latter is resisted by the plane DG, and DE is the force or velocity with which the body will begin to move down the plane DG. But $DE = DF \times \cos\theta = V\cos\theta$. Hence the body loses at D a portion of its velocity

$$= V - V\cos\theta = V(1 - \cos\theta) = V \text{ ver sin } \theta.$$

But if an angle or its arc is infinitely small, its sine is evidently also infinitely small, and its versed sine is an infinitely small quantity of the second order. For let AM (Fig. 2) be an infinitely small arc, Mp its sine, and Ap its versed sine. From the known properties of the circle, $Ap : pM = pM : pD$.

Fig. 2.



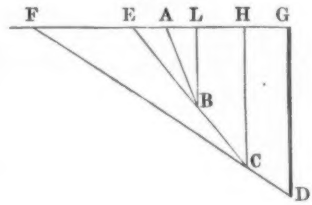
* The general mode of demonstration in this article has been taken from the French of MM. Allaize, Billy, Boudrot, and Puissant.

Now $p\,M$ is infinitely small with respect to $D\,p$; therefore $A\,p$ is infinitely small with respect to $p\,M$; that is, it is infinitely less than an infinitely small quantity, and is hence called an *infinitely small quantity of the second order*. Hence, if in the preceding figure θ is infinitely small, V ver sin θ may be regarded as zero, and the body suffers no diminution of velocity in passing from the plane $A\,D$ to the plane $D\,G$.

A curve may be regarded as made up of an infinite number of infinitely small straight lines, any two contiguous lines differing infinitely little in inclination; hence, if a body moves down a vertical curve, its retardation at each of the successive straight lines of which the curve is conceived to be composed will be an infinitely small quantity of the second order, and the sum of these infinitely small quantities of the second order will be an infinitely small quantity of the first order and may therefore be disregarded. The body, therefore, suffers no retardation in moving down a vertical curve.

Suppose (Fig. 3) a vertical curve composed of an infinite number of straight lines, $A\,B$, $B\,C$, $C\,D$, &c. Prolong the lines $B\,C$, $C\,D$, &c. until they meet the horizontal line $F\,G$ drawn through the point A. Suppose a heavy body to have descended the plane $A\,B$ to the point B. From the laws of descent down an inclined plane, it will then have the same velocity as if it had descended the plane $E\,B$, and since it suffers no diminution of velocity in passing to the plane $B\,C$, it will have, on reaching C, the same velocity as if it had descended the plane $E\,C$. In like manner, when the body reaches D, it will have the same velocity as if it had descended the plane $F\,D$. Moreover, from the laws referred to, the body will have at the points B, C, D, &c. the same velocity as if it had fallen freely through the vertical

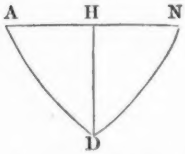
Fig. 3.



lines LB, HC, GD, &c. Hence, a body descending a vertical curve, in virtue of gravity, has at any point the same velocity as it would have acquired in falling freely through the vertical height of the curve passed over, and this velocity is independent of the nature of the curve.

When the body shall have reached the lowest point, D (Fig. 4), of the curve, it will, in consequence of inertia, begin to ascend the other branch, DN; and while it is passing over sides corresponding to those of which AD is conceived to be composed, gravity will gradually deprive it of those portions of velocity which were communicated to it in its descent along the branch AD. It will therefore ascend until it shall have reached a point, N, at the same height as the point A, from which it descended. At the point N it will begin to descend, and having reached D it will ascend the first branch to A, again to descend. The body will thus alternately move from A to N and from N to A. The movement from A to N, or from N to A, is called an *oscillation*.

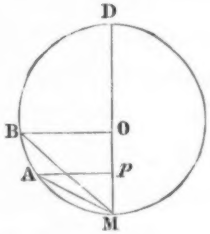
Fig. 4.



If the two branches of the curve ADN are symmetrical with respect to the vertical line HD all the corresponding elements of the two branches being equal, and being traversed with equal velocities, the times of describing them will be equal.

When the curve DBM (Fig. 5) is circular, the velocities acquired at M by two bodies that have descended the arcs AM and BM are to each other as the chords of these arcs. For, according to the laws of falling bodies and those of descent along inclined planes, these velocities are as the square roots of pM and OM, but these roots are to each other as the chords AM and BM. (See Davies's Legendre, B. IV. Prop. XXIII. Cor.)

Fig. 5.

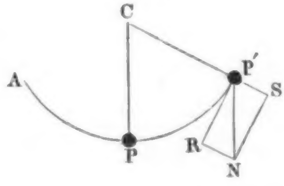


If we wish that the body, on arriving at M, should have a given velocity, V, we calculate the height $pM = \frac{V^2}{2g}$, and draw through the point p the horizontal line pA. This determines the point A, from which a body must descend along the arc AM, to acquire at M the velocity required.

2. A *simple pendulum* is a very small body of great density suspended by an exceedingly fine thread to a fixed point. Theoretically it is a mere point, supposed to be indued with weight, suspended by a thread without weight to a point about which it is perfectly free to revolve.

Suppose such a pendulum (Fig. 6) removed from its vertical position, CP, to the position CP', and then abandoned to gravity. Take $P'N$ in a vertical direction to represent the weight of the particle P. Resolve the force $P'N$ into the two forces $P'S$ in the direction of the thread, and $P'R$ in the direction of a tangent to the arc PP'. The former is resisted by the tension of the thread, and the latter produces motion down the curve. The pendulum will therefore be in the condition of a heavy body moving down the curve $P'P$, the tension of the thread corresponding to the resistance of the curve. Consequently it will move alternately from P' to A and from A to P'; and this motion would continue forever were it not for friction on the point of suspension, C, and the resistance of the air.

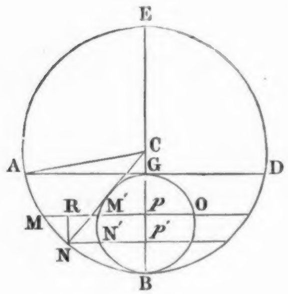
Fig. 6.



3. Considering a circular curve, EAB (Fig. 7), having its origin at E, as made up of infinitely small straight lines, let MN be one of these lines, and draw the perpendiculars Mp and Np' to the diameter passing through E. Draw also the radius CN to the point N, and RN perpendicular to Mp. The perpendiculars Mp and Np' are called *ordinates* of the curve, and pp' is called the *projection* of the

line MN upon the diameter EB. On account of the minuteness of the side MN, the radius CN may be regarded as perpendicular to it; therefore the triangles MNR and CNp' are similar, having the sides of the one perpendicular to those of the other. Hence $Np' : NR = NC : MN$, $\therefore MN = \frac{NC \times NR}{Np'} = \frac{NC \times p'p}{Np'}$. That is, any one of these infinitely small sides is equal to the product of its projection upon the diameter passing through the origin of the curve into the radius of the circle, divided by the ordinate corresponding to this side.

Fig 7.



4. It is now required to find the time of oscillation of a simple pendulum, whose length is known, through a circular arc of a very small number of degrees. Let ABD (Fig. 7) be this arc, and suppose that the pendulum, whose length is $l = CB$, starting from A, has reached the point M, and let u be the velocity it has acquired at this point. Draw the horizontal line AD, and upon GB as a diameter describe the circumference $GM'BO$. MN, as before supposed, is one of the infinitely small sides composing the curve AB, and $M'N'$ is one of those composing the curve $GM'B$. Let $Gp = x$, $Mp = y$, the small side $MN = s$, its projection pp', which is also the projection of $M'N'$, $= s'$, and the diameter GB, which is the height of the oscillation, $= d$. Also let t be the time in which the pendulum passes over MN, and T the time of an entire oscillation from A to D.

From the laws of falling bodies, and from what was proved in (1), $u = \sqrt{2gx}$; and since the minuteness of MN allows us to suppose that it is described uniformly with the velocity u, we have

$$t = \frac{MN}{u} = \frac{ls'}{y\sqrt{2gx}},$$

according to (3).

But $G\,B$, being the versed sine of a very small number of degrees, must itself be very small, and y, which is a mean proportional between $p\,B$ and $p\,E$, or $d-x$ and $2\,l-(d-x)$, may, without sensible error, be regarded as a mean between $d-x$ and $2\,l$, $\therefore\ y=\sqrt{2\,l(d-x)}$. Consequently

$$t=\frac{l\,s'}{\sqrt{4\,g\,l\,x\,(d-x)}}=\frac{\frac{1}{2}\,l\,s'}{\sqrt{gl\ x\,(d-x)}}=\frac{\sqrt{l}}{\sqrt{g}}\cdot\frac{\frac{1}{2}\,s'}{\sqrt{x\,(d-x)}}=\frac{\sqrt{l}}{d\sqrt{g}}\cdot\frac{\frac{1}{2}\,d\,s'}{\sqrt{x\,(d-x)}}.$$

Now $\sqrt{x(d-x)}=M'p$; hence, by (3), the factor $\frac{\frac{1}{2}\,d\,s'}{\sqrt{x\,(d-x)}}=M'N'$. Therefore $t=\frac{\sqrt{l}}{\sqrt{g}}\cdot\frac{M'\,N'}{d}$; and as we should find a similar result for each of the sides composing the arc $G\,M'\,B$, it follows that the time of descent through the height of this arc, that is, the time of descent through the arc $A\,M\,B$, or $\frac{1}{2}\,T=\frac{\sqrt{l}}{\sqrt{g}}\cdot\frac{G\,M'\,B}{G\,B}$, and $T=\frac{\sqrt{l}}{\sqrt{g}}\cdot\frac{G\,M'B\,O}{G\,B}$. But the last factor $=\pi$; $\therefore\ T=\frac{\pi\sqrt{l}}{\sqrt{g}}=\pi\sqrt{\frac{l}{g}}$. Hence the time of the oscillation of a simple pendulum through a very small portion of the circumference of a circle is equal to the ratio of the circumference of a circle to its diameter into the square root of the quotient arising from dividing the length of the pendulum by the force of gravity.

The value of T being independent of $d=G\,B$, it follows that, in very small portions of a circumference, the oscillations of a pendulum are *isochronous*, or the same in duration, whether the arc be greater or less, provided it does not exceed very narrow limits. The arcs must not be greater than four or five degrees.

Remark. — It may be proved by the higher mathematics that, if a pendulum oscillates in a curve called a *cycloid*, the time of oscillation will be perfectly independent of the length of the curve.

5. The time T' of oscillation of another pendulum whose length is l', in a place where the force of gravity is g', is $T'=\pi\sqrt{\frac{l'}{g'}}$;

$$\therefore\ T:T'=\sqrt{\frac{l}{g}}:\sqrt{\frac{l'}{g'}}.$$

Where the force of gravity is the same, that is, in equal latitudes, $T : T' = \sqrt{l} : \sqrt{l'}$. Hence, *In the same latitude, the times of oscillation of two pendulums differing in length are to each other as the square roots of the lengths.*

If $l = l'$, $T : T' = \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}} = \sqrt{g'} : \sqrt{g}$. Hence, *In different latitudes, the times of oscillation of two pendulums equal in length are inversely as the square roots of the different forces of gravity.*

When $T = T'$, $\sqrt{\frac{l}{g}} = \sqrt{\frac{l'}{g'}}$; $\therefore \frac{l}{g} = \frac{l'}{g'}$, or $l : l' = g : g'$. Hence, *The lengths of two pendulums which oscillate in equal times in different places are to each other as the intensities of gravity at those places.*

If T and T' are the times of oscillation of two pendulums at the same place, l and l' their lengths, and n and n' the number of oscillations which they make in the same given time, K, we have $K = nT = n'T'$; $\therefore n : n' = T' : T$. But $T' : T = \sqrt{l'} : \sqrt{l}$; $\therefore n : n' = \sqrt{l'} : \sqrt{l}$. Hence, *The numbers of oscillations made by two pendulums of different lengths in the same time, and at the same place, are to each other inversely as the square roots of the lengths.*

6. When the pendulum and the string or rod which connects it with the point of suspension have sensible masses, as in fact they always have, it is called a *compound pendulum.* But we have seen that the time of oscillation of a simple pendulum is less, the shorter the pendulum. Hence the material particles of a compound pendulum tend to vibrate more rapidly the nearer they are to the point of support. But, being connected by cohesion, all the particles of the pendulum must oscillate in the same time. Nevertheless, the tendency to rapid motion in the particles nearer to the point of support is counteracted by the tendency to a slower motion in the remoter particles, and the slower motion which these latter tend to have is accelerated by the tendency to a rapid motion in the former. There is, however, a particle, intermediate in position, which is neither

accelerated nor retarded, but which oscillates precisely as if it were unconnected with the other particles and constituted a simple pendulum. The point occupied by this particle is called the *centre of oscillation*, and the distance of this point from the point of suspension constitutes the virtual length of the pendulum.

If in the proportion $n : n' = \sqrt{l'} : \sqrt{l}$, which gives $l' = \frac{n^2 l}{n'^2}$, l, the length of a seconds pendulum, n, the number of oscillations it makes in any time, as one minute, are known, and we find by experiment the number of oscillations, n', which a compound pendulum makes in the same time, we can easily compute l', the virtual length of this compound pendulum.

By making a pendulum whose length, l, has been very accurately determined oscillate in a vacuum during a number, b, of seconds, and observing the number of oscillations, n, which it makes in this time, we shall have the time of one oscillation by dividing b by n; that is, $T = \frac{b}{n}$. Substituting this value of T in the equation $T = \pi\sqrt{\frac{l}{g}}$, we obtain $g = \frac{\pi^2 n^2 l}{b^2}$.

By careful experiments at London, Paris, New York, and other places, the values of g have been determined, either in the manner specified above or otherwise. At New York, the approximate value of g is 385.9134 inches = 32.15945 feet. This value, substituted in the formula $T = \pi\sqrt{\frac{l}{g}}$, in which T is made equal to one second, gives $l = \frac{g}{\pi^2} = 39.1012$ inches, the length of the seconds pendulum. At Paris the length is 39.12843 inches, and at London 39.13908 inches.

THE POLYTECHNIC SCHOOL AT PARIS.

History, Conditions for Admission, Courses of Lectures, Graphical Works, etc.

The Polytechnic School enjoys a renown so imposing, that it is admitted as an article of faith in France, as the most efficacious mode of procuring for the state distinguished and capable subjects, whether for the public service which it supplies, or for the advancement of the exact sciences.

In order to show the relation which it bears to the government I shall give a brief sketch of its history.

Before the Revolution there existed in France the following schools of application: *Ecole des Mines, Ecole des Ponts et Chaussées, Ecole de la Marine, Ecole du Génie Militaire*, and *Ecole de l'Artillerie*. These special schools were almost, if not wholly, disorganized by the political difficulties which grew out of the Revolution, and at the commencement of 1794 the Committee of Public Safety felt sensibly the need of filling by competent persons the numerous vacancies which were constantly occurring.

This committee was surrounded by men distinguished for their contributions to the exact sciences. These *savans*, who were also political men, embracing with ardor the principles of the Revolution, proposed, in order to satisfy the urgent needs of the moment, to create at Paris a school destined to replace in a measure the special schools, and at the same time to give an extraordinary extension to the sciences of Mathematics, Physics, and Chemistry.

Fourcroy, a celebrated chemist, was chosen from among their number to present to the Convention the project of a law for the creation of this school, which was designated by the name of Central School of Public Works (*Ecole Centrale des Travaux Publics*).

This was a few weeks after the fall of Robespierre, and Fourcroy, with considerable address, drew the attention of the assembly to the recent triumph, in saying, that the last conspirators had formed the project, not only for suppressing the arts and sciences, in order to march to power over the ruins of human knowledge, but also for crushing all the men and removing every means useful for instruction.

The Central School of Public Works was to supply the fine corps of engineers, and it was proposed to suppress the special schools when it should be firmly established. It was also to recommence the teaching of the exact sciences, which had been suspended by the Revolution, and to diffuse the taste for these sciences, by means of the students who returned home without entering the public service.

The number of students was to be four hundred, between the ages of sixteen and twenty.

They were to be taught Descriptive Geometry, the general principles of Analysis and their application to Descriptive Geometry, to the mechanics of solids and fluids, and to the calculation of the effect of machines; also Physics, Chemistry, Design, Civil Construction, Architecture, and Fortification.

The students were to receive salaries, as the greater part of the citizens would be unable to maintain their children at Paris for three years.

Fourcroy terminated his report in these words: "The committee must tell you that the greatness of this school is worthy of the people to whom it is presented, that it will be without a model in Europe, that it will satisfy both the needs of the republic, and those of general instruction, which we have felt for the past five years, that it will diffuse wider and wider through the whole republic the taste so advantageous for the exact sciences, and that it is finally one of the most powerful means to advance equally the useful arts and the human reason."

The Convention adopted the report, and provided the means for the immediate establishment of the institution; appointing a number of professors, among whom were LAGRANGE, MONGE, HACKETTE, PRONEY, and BERTHOLLET.

The opening of the courses took place on the 24th of May, 1795. The first lecture being given by LAGRANGE, on which occasion the founder mingled with the students to listen to the illustrious geometer. Arithmetic, which was the subject of the first lesson, says PRONEY, who was present, so dry and sterile as generally treated, acquired in the hands of LAGRANGE the elegance and fertility which characterize all his writings.

Unfortunately, often in the midst of their courses, the students who were not exempt from duty in the National Guard were obliged to assist in the defence of the Convention against the attacks of the Jacobins.

It is said that in these times the students were often in the greatest need of the necessaries of life, and were only relieved by the Convention, which caused to be distributed to one hundred and fifty of their number a pound of bread per day.

The following year the Convention changed the name to Polytechnic School, and at the same time organized the schools of application, and provided that hereafter they should only receive as students the graduates of the Polytechnic School. One of the provisions of this law which regulated the conditions for admission is curious, as it shows the spirit of the times; it was as follows: "No person can present himself as a candidate for admission unless he brings a certificate from the Municipality, attesting his good conduct, and that he has constantly manifested a love of liberty and equality and a hatred of tyranny."

The school suffered occasionally from the severity of the government toward those who were not Republicans. The minister annulled the examination of two candidates for not having satisfied the law just quoted.

"This just severity," he wrote, "will recall to the students the obligation which is imposed upon them to repay by their *civism*, as well as by their work, the instruction which the country furnishes. Thus it is the formal intention of the government to accord its favors and national distinctions only to those who render themselves worthy by their devotion to the republic."

A short time subsequent, it appeared expedient to give a striking token of adhesion to the political dogmas of the dominant party. The planting of a tree of liberty was one of the most august ceremonies of this new religion. It was therefore resolved, that on the day of the opening of the courses a tree of liberty should be planted within the enclosure of the school. They tried to obtain the presence of BONAPARTE, who was then at Paris; he promised to assist, but did not come. After the addresses of the professors, the assembly repaired to the court of the laboratories, where an Athenian poplar had just been planted. The director attached to it a tricolor, and they buried at its root an account of the inauguration; they chanted couplets and recited strophes filled with fire and enthusiasm; in short, nothing was forgotten which could electrify their souls. Unfortunately it began to rain, and the republican fervor of the students did not prevent their dispersing to seek a shelter in the study-rooms, whence they saw the concluding ceremonies from the windows. This incident marred the effect, and caused the director considerable vexation.

It was on another occasion, after LAGRANGE had announced the subject of his lectures for the following year to be the Theory of Functions, designed to remove the difficulties of the Differential Calculus, that the Minister of the Interior addressed the assemblage. After a short eulogy upon the professors and the school "just raised to the first rank in public instruction," he endeavored to make the students sensible of the gratitude which they owed their country, which "had sought them from their cradles, to put them in relation with the men that Europe honored most."

"If the love of country," said he, "acts as a *sentiment* upon the rest of mankind, it is permitted to think that to the *savans* the existence of this love is geometrically demonstrated. I can say, in the language which is familiar to you, liberty is the theorem given by nature, the republic is its demonstration, and patriotism is the corollary."

The limits of this sketch must cause me to pass over with the mere mention events which had an important influence upon the character of the school. These were, the resignation of LAGRANGE and the appointment of LACROIX as his successor; the return of BONAPARTE from Egypt, and, after the memorable day at Saint Cloud, the appointment of LAPLACE, then permanent examiner, as Minister of the Interior, and the election of LEGENDRE to fill the vacancy.

I cannot, however, refrain from giving an anecdote of the teaching of MONGE. I remember, says one of his pupils, that, after having explained the most difficult part of a geometrical figure (*épure*), he usually turned to his auditory in order to discover if they had comprehended him, and if he perceived that they had not, he would recommence the same explanation with new developments, and I have seen him thus repeat three times the same explanation.

CONDITIONS FOR ADMISSION THE PRESENT YEAR, 1861.

In order to be admitted to the school, the candidate must show that he is a French subject, or that he been naturalized, that he is between the ages of sixteen and twenty, that he is free from physical defect; and he must present also a diploma of Bachelor of Science, and must submit to two examinations upon the following subjects: —

1. Geometry, Plane, Solid, and Spherical.
2. Surveying, and the use of the ordinary instruments.
3. Algebra, including Progressions, Logarithms, Derived Functions, Theory of Equations, Theory of Differences and its application to the resolution of numerical equations and decomposition of rational fractions.
4. Trigonometry, Plane and Spherical.
5. Analytic Geometry of two and three dimensions, including the discussion of the general equation of three variables.
6. Descriptive Geometry.
7. Mechanics.
8. Physics, including Hydrostatics, Hydrodynamics, Statical Electricity, and Magnetism.
9. Inorganic Chemistry.
10. The French Language.
11. The German Language, in which the student is required to know the principal rules of the grammar, to translate an easy text, reply in German to simple questions, and also to present a German theme.
12. Drawing, in which the student is required to execute, — 1. A drawing in Descriptive Geometry; 2. A drawing after a model, and shaded in India ink; 3. A design in crayon.

The first of the examinations above mentioned serves to ascertain if the candidates have the knowledge required, and the second serves to determine their rank.

The school is under military discipline, and the students are allowed to leave the establishment only twice per week.

Each student pays per year the sum of one thousand francs.

COURSES OF LECTURES. — These are eleven in number.

I. *Course of Analysis.* — This embraces during the first year the Differential and Integral Calculus with their Applications to Algebra, Geometry, and Mechanics; during the second year, Definite Integrals, Integration of Differential Equations, Application of Analysis to Mechanics and Astronomy, and Calculus of Probabilities.

II. *Course of Descriptive Geometry and Stéréotomie.* — This includes, during the first year, Axiometric Perspective, Isometric Perspective, Perspective *Cavalière*, Conical Perspective, Representation of Shades and Shadows, Developable Surfaces, Warped Surfaces, Curvature of Surfaces, the Heliçoides, and Descriptive Geometry of one Plane.

During the second year, the Application to Wood-cutting (framing and construction of staircases), Stone-cutting (rampart arches, oblique arches, conical, warped, helicoidal, and compound arches).

III. *Course of Mechanics and Machines.* — This course for the first year is divided into three parts: — 1. Cinematics; 2. Statics; 3. Calculation of the Effect of Machines.

The course for the second year consists of, — 1. Dynamics; 2. Mechanics of Fluids; 3. Mechanics considered in motion.

IV. *Course of Physics.* — During the first year, Heat and Dynamical Electricity; during the second year, Acoustics and Optics.

V. *Chemistry.* — This embraces the treatise of Regnault, who is one of the professors.

VI. *Astronomy and Geodesy.*

VII. *Architecture and Public Works.* — The first includes the treatise of Reynard upon Architecture, and is divided into constructions in stone, in wood, and in iron, and the composition of edifices. The course upon Public Works consists of general notions of the construction of roads and bridges.

VIII. *Topography.*

IX. *Fortification and Military Art.*

X. *French Literature.*

XI. *The German Language.*

Graphical Works.

There are exercises and examples given to the students in every course, but there are regular detailed drawings required upon the following courses: —

I. *Descriptive Geometry.* — These consist of, — 1. The construction of the intersections of surfaces of the second degree; 2. The shading of cones traversed by cylinders in various ways; 3. One Plane Descriptive Geometry; 4. Linear Perspective, the drawing of a house, of an oblique bridge, and also the perspective of a *tone* with shades and shadows.

II. *Stereotomie.* — 1. Wood-cutting, including assemblages in wood, constructions of roofs, and winding staircases, 2. Stone-cutting, including construction of various doorways, oblique arches, ground and cloistered arches.

III. *Mechanics and Machines.* — The method of instruction in this branch deserves notice. A portion of a machine is placed before the students, and with pencil and paper only they are required to make a sketch of it; a measuring rule is then given them, and they are required to annex to each part of their sketch the real dimensions; afterward they are required to draw the object carefully and shade it. The students commence with elements as simple as a piston-rod with its articulations; afterwards they have toothed wheels, both cylindrical and conical, and finally a steam-engine.

IV. *Geodesy.* — The graphic works consist in making from given data a sun-dial; and 2. Delineation of maps.

V. *Military Art.* — Consisting of a trace of a front of Vauban, of Cormontaigne, and a project of attack upon a given place. The course is however very meagre, being designed to give those who are not to enter the military service hereafter some general notions which may be useful to them, — as, for instance, an engineer *des Ponts et Chaussées* may have occasion to lay out a road which borders upon a fortress, and for that it is necessary that he have some knowledge of this matter.

VI. *Architecture.* — Consisting of careful drawing of columns of the various orders, arcades, windows, porticos, and lastly the project of a public library.

VII. *Topographical Drawing.*

VIII. *Shading in India Ink.*

IX. *Portrait Drawing and Sketching of Landscapes.* — In each of the last three a model is placed before the student, and he has simply to copy it.

General Dispositions.

The lessons are usually an hour and a half in length; the last half-hour is set apart for questioning two of the students at the blackboard upon the subject of the lesson before the last.

On the table before the professor is an urn, containing one hundred and fifty balls, numbered according to the chairs in which the students sit; the professor selects at random a ball from the urn, and calls up the student occupying the chair corresponding to this number. The object of this is to find out who review the previous lesson before coming into the lecture-room.

The students are examined in each subject by the regularly appointed *repetiteurs* as often as once for every six lessons, and at the end of each half-year they are examined carefully over the whole ground by another set of men, regularly appointed, called *examinateurs*, and their rank is made up from the whole.

The students submit to military drill twice a week during the summer months.

The graduates have a choice, according to their rank, of the following services, viz.: That of *Mines*, *Ponts et Chaussées*, *Genie Maritime*, *Genie Militaire*, *Hydrographie*, *Lignes Telegraphiques*, *Commissariat de Marine*, *Tabac*, *Etat Major*, *Artillerie*, *Marine*.

To most of these services there are attached special schools, in which the students immediately enter, and pass three years in studying the specialty which they have chosen; and where there is no special school, as in the Tobacco service, the students study at the tobacco factory, and have regularly lectures and attend elsewhere to perfect themselves in the analysis of organic substances, and the higher departments of Chemistry. It is taken for granted, as the director of the studies said to me, that the students when they graduate are not fitted by their studies to enter upon any kind of practical service, and that they are only fitted in the best manner to commence the study of the application of the sciences of which they already know the principles, and hence the necessity for the schools of application; of several of these, particularly those which prepare for the civil service, I shall hereafter have occasion to speak in detail.

There are usually admitted about one hundred and seventy-five, and the number of applicants examined exceeds one thousand.

I might mention that the students are furnished with lithograph impressions of the lectures of the professors previous to their examinations, so that they are not absolutely dependent upon the notes which they take in the lecture-room, as is frequently the case in many of the other schools.

Note. — The award of the prizes for the solutions of problems in the March number, crowded out this month on account of the unexpected length of some of the articles, will be found in the next number.

Worcester's Quarto Dictionary The Standard

VERY SIGNIFICANT FACTS.

The following recommendations are from some of the most distinguished American and English scholars. They are but a few from many which have been received, testifying to the superiority of Worcester's Quarto Dictionary. These testimonials are of the highest value, for they have all been given during the present year, and after an examination of this work and of that which is endeavoring to hold the position of a rival. The scholars of America and of England, with scarcely an exception, have decided in favor of Worcester. Not a single scholar, equal in authority to any one mentioned below, can be cited as giving, after a comparison of the two works, the preference to Webster's Dictionary. We give the testimony: —

From C. C. Felton, LL.D., *President of Harvard College.*

Aware of the labor and care which had been devoted to this (the department of scientific terms) as well as to other parts of the work, I felt assured that Worcester's Quarto Dictionary would more nearly meet the public wants than any other hitherto published.

My expectations have been more than fulfilled. I find it not only rich beyond example in its vocabulary, but carefully elaborate in all the details, and thoroughly trustworthy as a guide to the most correct and elegant usage of the language.

From the Rev. Joseph Bosworth, D.D., F.R.S., *Professor of Anglo-Saxon, Oxford, England.*

It is the most complete and practical, the very best, as well as the cheapest English Dictionary that I know.

From George P. Marsh, LL.D.

The work of Dr. Worcester is unquestionably much superior to any other general dictionary of the language in every one of these particulars (orthography, pronunciation, definition, fulness of vocabulary, and precision and distinctness of definition).

From Rev. W. Whewell, D.D., *Master of Trinity College, England.*

I have repeatedly consulted the Dictionary since it has been in my possession, and have seen reason to think it more complete and exact than its predecessors.

From Charles Richardson, LL.D., *the oldest living English Lexicographer, England.*

I sincerely hope you may enjoy from your brethren, both in America and England, that tribute of honor to which you have earned so undoubted a title.

From D. R. Goodwin, D.D., *President of Trinity College, Hartford.*

It was but a short time since that I was led to commend another dictionary as, on the whole, and with some exceptions, the best and most complete thing of the kind within my knowledge. The commendation was honestly given at the time; but now *it must be withdrawn in favor of yours.* I consider your dictionary, in orthography, pronunciation, and definitions, as superior to any of its predecessors.

From Rev. W. B. Sprague, D.D., *of Albany, N. Y.*

My opinion of Worcester's Quarto Dictionary, after having given it as extended an examination as my circumstances would admit, is, that there is no other dictionary in the language that compares with it for completeness, accuracy, comprehensiveness, and precision, and perhaps I ought to add, that I have arrived at this conclusion rather contrary to a preconceived opinion.

From Rev. Henry A. Boardman, D D., *of Philadelphia.*

I particularly like it (the Dictionary), 1. Because of its very comprehensive character: 2 Because it adheres to the *settled orthography* of our noble language, — discarding those innovations which, however countenanced by certain publishing-houses, have never to any extent been accepted by the scholars of our country.

From Louis Agassiz, LL.D.

It is of great importance, when the nomenclature of science is gradually creeping into common use, that an English lexicon should embrace as much of it as is consistent with the language we speak. I am truly surprised and highly delighted to find you have succeeded far beyond my expectations in making the proper selection, and combining with it a remarkable degree of accuracy. More could hardly be given except in a scientific cyclopædia.

The following lines are quoted from Harper's Magazine for September. They serve to show very truthfully the comparative value of recent and old commendations: —

"Injustice. — Our attention has been called to a species of injustice of which publishers are sometimes guilty, in publishing commendations of school-books. without giving the dates when they were written. Especially does this merit reproval when these commendations are old, and when it is known that the writers have subsequently commended other and later publications in the same department. It will readily be seen that this is frequently not only an act of injustice to teachers who have had the courtesy to commend a book, but that it is also a fraud upon the public."

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THE NORTH AMERICAN REVIEW. No. CLXXXIX. — For October, 1860.

Contents. — Article I. Homer and his Heroines. II. Climatology. III. Life and Labors of Thomas Prince. IV. Edmund Waller. V. Lord Shaftesbury. VI. Second Volume of Palfrey's "History of New England." VII. Quarantine and Hygiene. VIII. Rush's Occasional Productions. IX. The English Language in America. X. The Origin of Species. XI. An "Inglorious Milton." XII. Critical Notices. XIII. New Publications.

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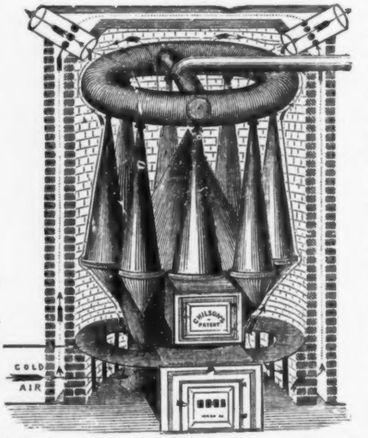

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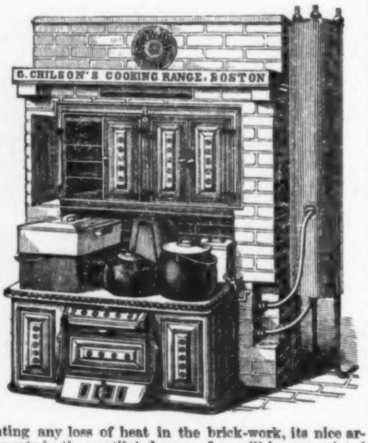

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